On the q-Laplace transform in the non-extensive statistical physics

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In this paper, q-Laplace transforms related to the non-extensive thermodynamics are investigated by using the algebraic operation of the non-extensive calculus. The deformed simple harmonic problem is discussed by using the q-Laplace transform.

I. INTRODUCTION

Boltzman-Gibbs statistical mechanics shows how fast microscopic physics with short-range interaction has as effect on much larger space-time scale. The Boltzman-Gibbs entropy is given by

$$S_{BG} = -k \sum_{i=1}^{W} p_i \ln p_i = k \sum_{i=1}^{W} \ln \frac{1}{p_i} \tag{1}$$

where k is a Boltzman constant, W is a total number of microscopic possibilities of the system and p_i is a probability of a given microstate among W different ones satisfying $\sum_{i=1}^{W} p_i = 1$. When $p_1 = \frac{1}{W}$, we have $S_{BG} = k \ln W$.

Boltzman-Gibbs theory is not adequate for various complex, natural, artificial and social system. For instance, this theory does not explain the case that a zero maximal Lyapunov exponent appears. Typically, such situations are governed by power-laws instead of exponential distributions. In order to deal with such systems, the non-extensive statistical mechanics is proposed by C.Tsallis [1,2]. The non-extensive entropy is defined by

$$S_q = k(\Sigma_i^W p_i^q - 1)/(1-q) \tag{2}$$

The non-extensive entropy has attracted much interest among the physicist, chemist and mathematicians who study the thermodynamics of complex system [3]. When the deformation parameter q goes to 1, Tsallis entropy (2) reduces to the ordinary one (1). The non-extensive statistical mechanics has been treated along three lines:

1. Mathematical development [4, 5, 6]
2. Observation of experimental behavior [7]
3. Theoretical physics (or chemistry) development [8]

The basis of the non-extensive statistical mechanics is q -deformed exponential and logarithmic function which is different from those of Jackson's [9]. The q-deformed exponential and q-logarithm of non-extensive statistical mechanics is defined by [10]

$$\ln_q t = \frac{t^{1-q} - 1}{1-q}, \quad (t > 0) \tag{3}$$

$$e_q(t) = (1 + (1-q)t)^{\frac{1}{1-q}}, \quad (x, q \in R) \tag{4}$$

where $1 + (1-q)t > 0$.

From the definition of q-exponential and q-logarithm, q-sum, q-difference, q-product and q-ratio are defined by [5, 6]

$$\begin{aligned} x \oplus y &= x + y + (1-q)xy \\ x \ominus y &= \frac{x-y}{1+(1-q)y} \end{aligned}$$

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$$\begin{aligned} x \otimes y &= [x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}} \\ x \oslash y &= [x^{1-q} - y^{1-q} + 1]^{\frac{1}{1-q}} \end{aligned} \tag{5}$$

It can be easily checked that the operation $\oplus$ and $\otimes$ satisfy commutativity and associativity. For the operator $\oplus$, the identity additive is 0, while for the operator $\otimes$ the identity multiplicative is 1. Indeed, there exist an analogy between this algebraic system and the role of hyperbolic space in metric topology [11]. Two distinct mathematical tools appears in the study of physical phenomena in the complex media which is characterized by singularities in a compact space [12].

For the new algebraic operation, q-exponential and q-logarithm have the following properties:

$$\begin{aligned} \ln_q(xy) &= \ln_q x \oplus \ln_q y \qquad e_q(x)e_q(y) = e_q(x \oplus y) \\ \ln_q(x \otimes y) &= \ln_q x + \ln_q y \qquad e_q(x) \otimes e_q(y) = e_q(x+y) \\ \ln_q(x/y) &= \ln_q x \ominus \ln_q y \qquad e_q(x)/e_q(y) = e_q(x \ominus y) \\ \ln_q(x \oslash y) &= \ln_q x - \ln_q y \qquad e_q(x) \oslash e_q(y) = e_q(x-y) \end{aligned} \tag{6}$$

From the associativity of $\oplus$ and $\otimes$, we have the following formula :

$$\underbrace{t \oplus t \oplus t \oplus \cdots \oplus t}_{n\ times} = \frac{1}{1-q}\{[1+(1-q)t]^n - 1\} \tag{7}$$

$$t^{\otimes^n} = \underbrace{t \otimes t \otimes t \otimes \cdots \otimes t}_{n\ times} = [nt^{1-q} - (n-1)]^{\frac{1}{1-q}} \tag{8}$$

II. Q-LAPLACE TRANSFORM

In this section, we find the q-Laplace transform related to the non-extensive thermodynamics. From the relation

$$e_q(x)e_q(y) = e_q(x \oplus y), \tag{9}$$

the q-analogue of e^{nx}, $(n \in Z)$ is given by

$$e_q(n \odot t) = [e_q(t)]^n = e_q\left(\frac{1}{1-q}[(1+(1-q)t)^n - 1]\right) = (1+(1-q)t)^{\frac{n}{1-q}} \tag{10}$$

Then we have

$$e_q(0 \odot t) = 1 \tag{11}$$

and the inverse of $e_q(n \odot t)$ is $e_q((-n) \odot t)$. For this adoption, q- Laplace kernel is defined by

$$e_q((-s) \odot t) = [e_q(t)]^{-s} = e_q\left(\frac{1}{1-q}[(1+(1-q)t)^{-s} - 1]\right) = (1+(1-q)t)^{-\frac{s}{1-q}} \tag{12}$$

Therefore q-Laplace transform is defined by

$$L_s(F(t)) = \int_0^\infty [e_q(t)]^{-s} F(t)dt, \quad (s > 0) \tag{13}$$

Limiting $q \to 1$, the eq.(13) reduces to an ordinary Laplace transform. Form now on we assume that s is sufficinetly large.

Since, for two functions $F(t)$ and $G(t)$, for which the integral exist

$$L_s(aF(t) + bG(t)) = aL_s(F(t)) + bL_s(G(t)), \tag{14}$$

the q-Laplace transform is linear.

For $F(t) = t^N$, $(N = 0, 1, 2, \cdots)$, we have the following result.

Theorem 1 *For sufficiently large s , when $q<1$, the following holds:*

$$L_s(t^N)=\frac{N!}{(s;1-q)_{N+1}} \tag{15}$$

where

$$(a;Q)_n=\begin{cases}1 & (n=0\ )\\ \prod_{k=1}^{n}(a-kQ) & (n\geq 1)\end{cases} \tag{16}$$

Proof. Let us assume that the eq.(15) holds for t^N. Then,

$$\begin{aligned} L_s(t^{N+1}) &= \int_0^\infty [e_q(t)]^{-s}t^{N+1}dt \\ &= \int_0^\infty (1+(1-q)t)^{-\frac{s}{1-q}}t^{N+1}dt \\ &= \left[\frac{(1+(1-q)t)^{-\frac{s}{1-q}+1}}{1-q-s}t^{N+1}\right]_0^\infty + \frac{N+1}{s-(1-q)}\int_0^\infty (1+(1-q)t)^{-\frac{s-(1-q)}{1-q}}t^N dt \\ &= \frac{N+1}{s-(1-q)}L_{s-(1-q)}(t^{N+1}) \\ &= \frac{(N+1)N!}{(s-(1-q))(s-(1-q);1-q)_{N+1}} \\ &= \frac{(N+1)!}{(s;1-q)_{N+2}} \end{aligned} \tag{17}$$

In a similar way, we can obtain the q-Laplace transform for $e_q(a\odot t)$ which is given by

$$e_q(a\odot t)=[e_q(t)]^a$$

Theorem 2 *For sufficiently large s , when $q<1$, the following holds:*

$$L_s(e_q(a\odot t))=\frac{1}{s-a-(1-q)} \tag{18}$$

Proof. It is trivial.

In order to obtain q-Laplace transform for the trigonometric function, we need q-analogue of Euler identity. The q-Euler formula is given by

$$e_q(ia\odot t)=C_q(a\odot t)+iS_q(a\odot t), \tag{19}$$

where q-cosine and q-sine functions are defined by

$$C_q(a\odot t)=\cos(\frac{a}{1-q}\ln(1+(1-q)t))$$

$$S_q(a\odot t)=\sin(\frac{a}{1-q}\ln(1+(1-q)t)) \tag{20}$$

and we used the following identity.

$$p^i=e^{i\ln p}=\cos\ln p+i\sin\ln p \tag{21}$$

Indeed, q-cosine and q-sine functions can be expressed in terms of q-exponential as follows:

$$C_q(a \odot t) = \frac{1}{2}[e_q(ia \odot t) + e_q((-ia) \odot t)$$

$$S_q(a \odot t) = \frac{1}{2i}[e_q(ia \odot t) - e_q((-ia) \odot t) \tag{22}$$

Then we have the q-Laplace transform for q-sine and q-cosine functions :

Theorem 3 *For sufficiently large s , when $q < 1$, the following holds:*

$$L_s(C_q(a \odot t)) = \frac{s-(1-q)}{(s-(1-q))^2 + a^2}$$

$$L_s(S_q(a \odot t)) = \frac{a}{(s-(1-q))^2 + a^2} \tag{23}$$

Proof. It is trivial.
The eq.(23) can be written in terms of the ordinary sine and cosine functions :

$$L_s(\cos[\frac{a}{1-q}\ln(1+(1-q)t)]) = \frac{s-(1-q)}{(s-(1-q))^2 + a^2}$$

$$L_s(\sin[\frac{a}{1-q}\ln(1+(1-q)t)]) = \frac{a}{(s-(1-q))^2 + a^2} \tag{24}$$

The q-sine function and q-cosine function have the following zeros:

$$S_q(1 \odot t_n) = S_q(t_n) = 0, \quad C_q(1 \odot u_n) = C_q(u_n) = 0, \tag{25}$$

where

$$t_n = \ln_q e^{n\pi}, \quad u_n = \ln_q e^{(n+\frac{1}{2})\pi}, \quad n \in Z \tag{26}$$

From the zeros of the q-sine function and q-cosine function, we have the following Theorem:

Theorem 4 *For sufficiently large s , when $q < 1$, the following holds:*

$$S_q(t) = t\prod_{j=1}^{\infty}\left(1 - \frac{t}{\ln_q e^{j\pi}}\right)\left(1 - \frac{t}{\ln_q e^{-j\pi}}\right) \tag{27}$$

$$C_q(t) = \left(1 - \frac{t}{\ln_q e^{-\pi/2}}\right)\prod_{j=1}^{\infty}\left(1 - \frac{t}{\ln_q e^{(j+1/2)\pi}}\right)\left(1 - \frac{t}{\ln_q e^{-(j+1/2)\pi}}\right) \tag{28}$$

Proof. From the zeros of the q-sine function , we can set

$$\frac{S_q(t)}{t} = A\prod_{j=1}^{\infty}\left(1 - \frac{t}{\ln_q e^{j\pi}}\right)\left(1 - \frac{t}{\ln_q e^{-j\pi}}\right)$$

Because $\lim_{t\to 0}\frac{S_q(t)}{t}=1$, we have $A=1$, which proves the eq.(27). Similarly we can easily prove the eq.(28).
Theorem 4 can be also written as follows :

Theorem 5 *For sufficiently large s , when $q<1$, the following holds:*

$$S_q(t)=t\prod_{j=1}^{\infty}\left[1+(1-q)t-\left(\frac{(1-q)t}{2\sinh(\frac{j(1-q)\pi}{2})}\right)^2\right] \tag{29}$$

$$C_q(t)=\left(1-\frac{t}{\ln_q e^{-\pi/2}}\right)\prod_{j=1}^{\infty}\left[1+(1-q)t-\left(\frac{(1-q)t}{2\sinh(\frac{1}{2}(1-q)(j+1/2)\pi)}\right)^2\right] \tag{30}$$

Proof. It is trivial from the formula $\cosh x-1=2\sinh^2\frac{x}{2}$.

III. Q-LAPLACE TRANSFORM AND DIFFERENTIAL EQUATION

Now we discuss the q- differential equation. The main purpose of q-Laplace transform is in converting q-differential equation into simpler forms which may be solved more easily. Like the ordinary Laplace transform , we can compute the q-Laplace transformation of derivative by using the definition of the q-Laplace trnsform, which is given by

$$L_s(F'(t))=sL_{s+1-q}(F(t))-F(0) \tag{31}$$

An extension gives

$$L_s(F''(t))=s(s+1-q)L_{s+2(1-q)}(F(t))-sF(0)-F'(0) \tag{32}$$

Generally, we have following theorem :

Theorem 6 *For sufficiently large s , when $q<1$, the following holds:*

$$L_s(F^{(n)}(t))=[s;1-q]_nL_{s+n(1-q)}(F(t))-\sum_{i=0}^{n-1}[s;1-q]_{n-1-i}F^{(i)}(0), \tag{33}$$

where $F^{(0)}(0)=F(0)$ and

$$[a;Q]_n=\begin{cases}1 & (n=0\ )\\ \prod_{k=1}^{n}(a+kQ) & (n\geq 1)\end{cases} \tag{34}$$

Proof. Let us assume that the eq.(34) holds for n. Then,

$$\begin{aligned}L_s(F^{(n+1)}(t)) &= \int_0^{\infty}(1+(1-q)t)^{-\frac{s}{1-q}}F^{(n+1)}(t)dt\\ &= sL_{s+1-q}(F^{(n)}(t))-F^{(n)}(0)\\ &= s\{[s+1-q;1-q]_nL_{s+(n+1)(1-q)}(F(t))-\sum_{i=0}^{n-1}[s+1-q;1-q]_{n-1-i}F^{(i)}(0)\}-F^{(n)}(0)\\ &= [s;1-q]_{n+1}L_{s+(n+1)(1-q)}(F(t))-\sum_{i=0}^{n}[s;1-q]_{n-i}F^{(i)}(0)\end{aligned} \tag{35}$$

We have another formula for the q-Laplace transform of derivative as follows:

$$L_s(F'(t)) = sL_s\left(\frac{F(t)}{1+(1-q)t}\right) - F(0) \tag{36}$$

An extension gives

$$L_s(F''(t)) = s(s+1-q)L_s\left(\frac{F(t)}{(1+(1-q)t)^2}\right) - sF(0) - F'(0) \tag{37}$$

Generally, we have the following theorem:

Theorem 7 *For sufficiently large s , when $q < 1$, the following holds:*

$$L_s(F^{(n)}(t)) = [s;1-q]_n L_s\left(\frac{F(t)}{(1+(1-q)t)^n}\right) - \sum_{i=0}^{n-1}[s;1-q]_{n-1-i}F^{(i)}(0) \tag{38}$$

Proof. It is not hard to prove Theorem 7.
Comparing Theorem 6 with Theorem 7, we have the following Lemma:

Lemma 8 *For sufficiently large s , when $q < 1$, the following holds:*

$$L_s\left(\frac{F(t)}{(1+(1-q)t)^n}\right) = L_{s+n(1-q)}(F(t)) \tag{39}$$

Proof. It is trivial.
With the help of q-Laplace transform of the derivative, we can solve some differential equation. It is worth noting that $e_q(t)$ is not invariant under the ordinary derivative, instead it obeys

$$\frac{d}{dt}(e_q(t)) = \frac{1}{1+(1-q)t}e_q(t) \tag{40}$$

Now consider the following differential equation:

$$F'(t) = \frac{F(t)}{1+(1-q)t}, \quad F(0) = 1 \tag{41}$$

It is evident that $e_q(t)$ is a solution of the eq.(40).
Let us consider the vertical motion of a body in a resisting medium in which there again exists a retarding force proportional to the velocity. Let us consider that the body is projected downward with zero initial velocity $v(0) = 0$ in a uniform gravitational field. The equation of motion is then given by

$$m\frac{d}{dt}v = mg - kv(t) \tag{42}$$

This equation is not solved by using the q-Laplace transform , instead we solve the following equation :

$$m\frac{d}{dt}v = mg - k\frac{v(t)}{1+(1-q)t} \tag{43}$$

The solution of the eq.(43) is then given by

$$v(t) = \frac{g}{1-q+\frac{k}{m}}(1+(1-q)t - [e_q(t)]^{-\frac{k}{m}}) \tag{44}$$

Similarly, we can modify the harmonic problem whose equation of motion is given by

$$m\left(\frac{d}{dt}\right)^2 x(t) = -mw^2\frac{x(t)}{(1+(1-q)t)^2}, \tag{45}$$

where $x(0)=A$, $\left(\frac{d}{dt}x\right)(0)=0$. The solution of the eq.(45) is then given by

$$x(t) = A(1+(1-q)t)^2\left\{C_{\frac{3-q}{2}}\left(\sqrt{w^2-(\frac{q-1}{2})^2}\odot t\right)+\frac{q-1}{\sqrt{w^2-(\frac{q-1}{2})^2}}S_{\frac{3-q}{2}}\left(\sqrt{w^2-(\frac{q-1}{2})^2}\odot t\right)\right\} \tag{46}$$

The eqs.(43)and (45) seem to be too artificial due to the factor $1+(1-q)t$. Instead, we can introduce the q-derivative [6] instead of the ordinary time derivative as follows:

$$D_tF(t) = \lim_{s\to t}\frac{F(t)-F(s)}{t\ominus s} = [1+(1-q)t]\frac{dF}{dt}$$

The Leibniz rule for q-derivative is as follows:

$$D_t[F(t)G(t)] = D_t[F(t)]G(t)+F(t)D_t[G(t)] \tag{47}$$

Then the eq.(43) is replaced as follows :

$$mD_tv = mg-kv(t) \tag{48}$$

The solution of the eq.(48) is then given by

$$v(t) = \frac{mg}{k}(1-[e_q(t)]^{-\frac{k}{m}}) \tag{49}$$

Similarly, the eq.(45) is replaced as follows:

$$mD_t^2x(t) = -mw^2x(t) \tag{50}$$

Using the q-Laplace transform, we get the solution of the eq.(50) :

$$x(t) = AC_q(w\odot t) \tag{51}$$

Obtaining these solutions, we used the following theorem:

Theorem 9 *For sufficiently large s , when $q<1$, the following holds:*

$$L_s((1+(1-q)t)^nF^{(n)}(t)) = [s-(n+1)(1-q);1-q]_nL_s(F(t))-\sum_{i=0}^{n-1}[s-(n+1)(1-q);1-q]_{n-1-i}F^{(i)}(0) \tag{52}$$

Proof. Let us assume that the eq.(52) holds for n. Then,

$$\begin{aligned}L_s((1+(1-q)t)^nF^{(n+1)}(t)) &= \int_0^\infty(1+(1-q)t)^{-\frac{s}{1-q}+n+1}F^{(n+1)}(t)dt\\ &= (s-(n+1)(1-q))L_s((1+(1-q)t)^nF^{(n)}(t))-F^{(n)}(0)\\ &= (s-(n+1)(1-q))\{[s-(n+1)(1-q);1-q]_nL_s(F(t))\\ &- \sum_{i=0}^{n-1}[s-(n+1)(1-q);1-q]_{n-1-i}F^{(i)}(0)\}-F^{(n)}(0)\\ &= [s-(n+2)(1-q);1-q]_{n+1}L_s(F(t))\\ &- \sum_{i=0}^{n}[s-(n+2)(1-q);1-q]_{n-i}F^{(i)}(0)\end{aligned} \tag{53}$$

The eq.(50) is also obtained by using the variational method whose Lagrangian is given by

$$L = \int dt \left(\frac{1}{2}(D_t x)^2 - U(x) \right) \tag{54}$$

The equation of motion is then given by

$$D_t \left(\frac{\partial L}{\partial (D_t x)} \right) - \frac{\partial L}{\partial x} = 0, \tag{55}$$

where the momentum p is defined by

$$p = \frac{\partial L}{\partial (D_t x)} \tag{56}$$

For the harmonic potential $U = \frac{1}{2} m w^2 x^2$, we have the eq.(50). This equation is rewritten by

$$(1 + (1-q)t)^2 \ddot{x} + (1-q)(1 + (1-q)t)\dot{x} = -w^2 x \tag{57}$$

Replacing

$$\eta = \frac{1}{1-q} \ln(1 + (1-q)t), \tag{58}$$

the eq.(57) is then as follows :

$$\frac{\partial^2 x}{\partial \eta^2} = -w^2 x(\eta) \tag{59}$$

Solving the eq.(59), we have

$$x(t) = A \cos w\eta = A C_q(w \odot t) \tag{60}$$

Here, let us investigate the times t_n when a body goes back to the initial position. This time is determined by

$$t_n = \ln_q e^{\frac{2\pi n}{w}}, \quad (t = 0, 1, 2, \cdots) \tag{61}$$

When $q < 1$ and $w > 0$, we have the following inequality:

$$t_{n+1} - t_n > t_n - t_{n-1} \tag{62}$$

Thus, the time when a body goes back to the initial position keeps increasing.

IV. CONCLUSION

In this paper, we used the algebraic operation and differential calculus related to the non-extensive thermodynamics to investigate the q-Laplace transform. We used the q-Laplace transform to solve some differential equation such as harmonic oscillator problem.

We think that this work will be applied to some q-differential equation which might appear in the study of the non-extensive statistical mechanics. We hope that these work and their related topics will be clear in the near future.

Refernces

[1] C.Tsallis, J.Stat.Phys. **52** (1988) 479.
[2] E.Curado, C.Tsallis, J.Phys. **A 24** (1991) L69.
[3] A.Cho, Science **297** (2002) 1268.
[4] S. Plastino, Science **300** (2003) 250.
[5] L.Nivanen, A.Le Mehaute , Q.Wang, Rep.Math.Phys.**52** (2003) 437
[6] E. Borges, Physica **A 340** (2004) 95.
[7] S.Abe, A.Rajagopal, Science **300** (2003) 249.
[8] V.Latora, A,Rapisarda, A.Robledo, Science **300** (2003) 250.
[9] F. Jackson, Mess.Math. **38** , 57 (1909).
[10] C.Tsallis, Quimica Nova **17** (1988) 479.
[11] A.Beardon, An Introduction to hyperbolic geometry, Oxford University Press, New York, (1991).
[12] A.Mehaute, R.Nigmatullin, L.Nivanen, Fleches du temps et geometrie fractale. Hermes, Paris, (1998).